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Fluctuation induced reconstruction of phase transition

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Abstract. Within mean field approximation, a procedure is elaborated to consider noise induced phase transitions with arbitrary relations between the noises of different degrees of freedom. The proposed approach is applied to investigate effects of cross correlation between noises in the generalized synergetic model of Lorenz type. This cross correlation is shown to induce phase transitions of the dynamical system under consideration. Additionally, we find the correlation between noises transforms a synergetic behavior to a thermodynamic one.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise and Brownian motion – 47.20.Ky Nonlinearity (including bifurcation theory)

Introduction

The prominent effect of noise on systems far off equilibrium attracted much attention from scientists in the last few decades. There are many physical situations in which the noise exhibits a constructive role in the behavior of nonlinear systems. Recently, noise was demonstrated to lead to a host of new amazing phenomena in such systems: noise induced unimodal-bimodal transitions in zero-dimensional models [1], noise induced phase transitions with a symmetry and ergodicity breaking in extended systems [2–5], stochastic resonance [6,7], noise induced pattern formation [8] (for review, see [4] and citations therein).

A lot of attention has been devoted currently to the study of reentrant noise induced phase transitions where the ordered state is characterized by a nonzero order parameter and exists only inside a window of control parameters, noise intensities and coupling constant. Recently, the ordering of the system has been shown to be the consequence of the interplay between the noise, the spatial coupling and the nonlinearity [3,5,9,10]. In particular, if at every single site a noise generates a short time instability, a spatial coupling can derive to a non-trivial stable state. In such a situation, naive predictions based on a deterministic analysis appear to be far from reliable.

Usually, for the sake of simplicity, only special models are considered either of additive noise or multiplicative noise with intensity in the form of linear function and with bare x^m-potential ($m > 3$) [4]. Suitable choice of deterministic drift and noise amplitude is shown to lead to both continuous and discontinuous bifurcations which are associated with the second or first order phase transitions, respectively. It is in good correspondence with the Landau theory of phase transitions where a form of the free energy potential defines the order of phase transition. Much more complicated approaches have been implemented to consider both white and coloured noises but cross correlations between different fluctuations do not take into account [2,3,5].

An interesting problem, which is treated in our paper, is understanding the role of such a cross correlation in stochastic systems: understanding whether they exert an influence on phase transitions or not. Usually, one holds the opinion that cross correlation introduces only weak corrections to results obtained for uncorrelated fluctuations. In this work, we wish to show the crucial role of cross correlation between fluctuations and show that they can lead to change in the order of phase transitions. Because such a problem can not be solved correctly by using the standard methods based on the Novikov theorem [11] or the unified coloured noise approximation [5,12], we derive a clear scheme to account for the above correlations on the basis of the cumulant expansion method proposed by Van Kampen [13] and developed in reference [14]. As a result of this, a perturbation theory is constructed with a small parameter being an auto or cross correlation time of different noise. The derived procedure is applied to a three-parameter Lorenz system with correlated noises. Such correlation is shown to lead the dynamical system to a chain of phase transitions being relevant rather to a thermodynamic picture than a synergetic one that is related to uncorrelated fluctuations.

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Approximation method

Let us focus on an extended system where evolution of physical quantity is given by the Langevin equation

$$\frac{\partial x(\mathbf{r},t)}{\partial t} = f(x) + D\Delta x + \sum_\mu g_\mu(x)\xi_\mu(\mathbf{r},t), \quad (1)$$

here the index μ denotes number of Langevin forces acting with amplitudes $g_\mu(x)$; $f(x) = -\partial V/\partial x$ is the force that drives the system which is defined with the help of bare potential $V(x)$; the effect of spatial coupling with the constant D is presented through the Laplacian $\Delta \equiv \partial^2/\partial\mathbf{r}^2$. In many physical systems $x = x(\mathbf{r},t)$ is a coarse-grained field, representing the local density of a relevant physical variable (relative concentration in a binary alloy, magnetization, etc.). As usual, a disordered state (phase) corresponds to $x(\mathbf{r},t) = 0$ (homogeneous mixture in alloys), an ordered state is given by $x(\mathbf{r},t) \neq 0$.

Considering a stochastic system with more than one noise one has to deal with an effect of correlation between noises. Let us assume that noises are Gaussian distributed, white in space with zero mean and

$$\langle \xi_\mu(\mathbf{r},t)\xi_\nu(\mathbf{r}',t')\rangle = \delta(\mathbf{r}-\mathbf{r}')C_{\mu,\nu}(t-t'), \quad (2)$$

here for correlation functions one has $C_{\mu,\nu}(t,t') = C_{\nu,\mu}(t,t')$.

To explore the picture of noise induced phase transitions one needs to determine the order parameter $\eta = \langle x\rangle$ where angle brackets denote an averaging over probability density $P(x,t)$. Therefore, we need to find the distribution function $P(x,t)$ at first. For this purpose we represent the system in the regular d-dimension lattice of mesh size $\Delta l = 1$

$$\dot{x}_i = f_i + \frac{D}{2d}\sum_j \widehat{D}_{ij}x_j + \sum_\mu g_{\mu i}\xi_{\mu i}(t), \quad (3)$$

where $f_i = f(x_i)$, $g_{\mu i} = g_\mu(x_i)$; here the definition of the Laplacian operator on the grid is used

$$\Delta \to \sum_j \widehat{D}_{ij} = \sum_{j\in nn(i)} (\delta_{nn(j)} - 2d\delta_{ij}). \quad (4)$$

To construct an equation for the probability density we exploit a conventional device and proceed from the continuity equation

$$\frac{\partial}{\partial t}\rho(\{x_i\},t) = -\sum_i \frac{\partial}{\partial x_i}(\dot{x}_i\rho(\{x_i\},t)). \quad (5)$$

The probability density function is given by the averaging over noise, *i.e.* $P(\{x_i\},t) = \langle\rho(\{x_i\},t)\rangle$. Inserting the time derivative from equation (3) into equation (5) we have

$$\frac{\partial}{\partial t}\rho(\{x_i\},t) = -\sum_i \left[\widehat{\mathcal{L}}_i - \frac{\partial}{\partial x_i}\sum_\mu g_{\mu i}\xi_{\mu i}\right]\rho(\{x_i\},t), \quad (6)$$

where

$$\widehat{\mathcal{L}}_i = \frac{\partial}{\partial x_i}\left(f_i + \sum_j \widehat{D}_{ij}x_j\right). \quad (7)$$

In the interaction representation

$$\wp = \sum_i \mathrm{e}^{\widehat{\mathcal{L}}_i t}\rho. \quad (8)$$

Equation (6) is reduced to the form

$$\frac{\partial}{\partial t}\wp = -\sum_i \mathrm{e}^{\widehat{\mathcal{L}}_i t}\frac{\partial}{\partial x_i}\sum_\mu g_{\mu i}\xi_{\mu i}\mathrm{e}^{-\widehat{\mathcal{L}}_i t}\wp \equiv \sum_i\sum_\mu \epsilon_\mu \mathcal{R}_\mu(x_i,t)\wp, \quad (9)$$

where the corresponding small parameter ϵ_μ measures a value of fluctuations defined through the noise intensity and correlation scale. A standard and effective device to solve such a type of stochastic equation is the well-known cumulant expansion method, developed by Van Kampen [13]. Neglecting terms of the order $O(\epsilon^3)$, in the main approximation we get the following kinetic equation:

$$\frac{\partial}{\partial t}\langle\wp\rangle = \sum_i\sum_{\mu,\nu}\epsilon_\mu\epsilon_\nu\int_0^t \langle\mathcal{R}_\mu(x_i,t)\mathcal{R}_\nu(x_i,t')\rangle\langle\wp\rangle\mathrm{d}t'. \quad (10)$$

In the original representation for the probability density $P(\{x_i\},t)$ equation (10) reads

$$\frac{\partial}{\partial t}P(\{x_i\},t) = \sum_i\Bigg[-\widehat{\mathcal{L}}_i P(\{x_i\},t) + \frac{\partial}{\partial x_i}\sum_{\mu,\nu} g_{\mu i}\int_0^t C_{\mu\nu}(t,t')\mathrm{e}^{-\widehat{\mathcal{L}}_i(t-t')}\frac{\partial}{\partial x_i}g_{\nu i}P(\{x_i\},t')\mathrm{d}t'\Bigg]. \quad (11)$$

For $t \gg \tau_\mu$, where τ_μ is the corresponding correlation scale, we can put $\langle\rho(\{x_i\},t')\rangle = \langle\rho(\{x_i\},t)\rangle$ and the upper limit of the integration taken, ∞. This yields

$$\frac{\partial}{\partial t}P = \sum_i\Big[-\widehat{\mathcal{L}}_i + \sum_{\mu,\nu}\widehat{L}^{(0)}_{\mu i}\int_0^\infty C_{\mu\nu}(t,t-\tau)\mathrm{e}^{-\widehat{\mathcal{L}}_i t}\widehat{L}^{(0)}_{\nu i}\mathrm{e}^{\widehat{\mathcal{L}}_i t}\mathrm{d}\tau\Big]P, \quad (12)$$

here we use a notation

$$\widehat{L}^{(0)}_{\mu i} \equiv \frac{\partial}{\partial x_i}g_{\mu i} = \frac{\partial}{\partial x_i}g_\mu(x_i) \quad (13)$$

and put $P \equiv P(\{x_i\},t)$. Expanding exponents one arrives at the following perturbation expansion

$$\frac{\partial}{\partial t}P = \sum_i\Bigg[-\frac{\partial}{\partial x_i}\left(f_i + \sum_j \widehat{D}_{ij}x_j\right) + \sum_{n=0}^\infty\sum_{\mu,\nu}\widehat{L}^{(0)}_{\mu i}C^{(n)}_{\mu\nu}\widehat{L}^{(n)}_{\nu i}\Bigg]P, \quad (14)$$

where moments of the correlation function are

$$C_{\mu\nu}^{(n)} = \frac{1}{n!}\int_0^\infty \tau^n C_{\mu\nu}(t,t-\tau)\mathrm{d}\tau. \qquad (15)$$

Operators $\widehat{L}_\nu^{(n)}$ are defined through the commutator

$$\widehat{L}_{\mu i}^{(n)} = \left[\widehat{L}_{\mu i}^{(n-1)}, \widehat{\mathcal{L}}_i\right]. \qquad (16)$$

The first order approximation yields

$$\begin{aligned}\widehat{L}_{\mu i}^{(1)} =& \frac{\partial}{\partial x_i} g_{\mu i}\frac{\partial}{\partial x_i}\left(f_i + \sum_j \widehat{D}_{ij}x_j\right)\\ &- \frac{\partial}{\partial x_i}\left(f_i + \sum_j \widehat{D}_{ij}x_j\right)\frac{\partial}{\partial x_i}g_{\mu i}. \qquad (17)\end{aligned}$$

According to obtained expressions the zero-order contribution of correlations gives the expected terms for the Kramers-Moyal expansion [15]

$$\begin{aligned}\mathcal{D}_1^{(0)} &= \sum_{\mu,\nu} C_{\mu\nu}^{(0)} g_{\mu i}\frac{\partial g_{\nu i}}{\partial x_i},\\ \mathcal{D}_2^{(0)} &= \sum_{\mu,\nu} C_{\mu\nu}^{(0)} g_{\mu i} g_{\nu i}; \qquad (18)\end{aligned}$$

a contribution of first order terms gives

$$\begin{aligned}\mathcal{D}_1^{(1)} =& \sum_{\mu,\nu} C_{\mu\nu}^{(1)}\frac{\partial g_{\nu i}}{\partial x_i}\left[\left(f_i + \sum_j \widehat{D}_{ij}x_j\right)\frac{\partial g_{\mu i}}{\partial x_i}\right.\\ &\left. + g_{\mu i}\frac{\partial}{\partial x_i}\left(f_i + \sum_j \widehat{D}_{ij}x_j\right)\right],\\ \mathcal{D}_2^{(1)} =& \sum_{\mu,\nu} C_{\mu\nu}^{(1)} g_{\nu i}\left[\left(f_i + \sum_j \widehat{D}_{ij}x_j\right)\frac{\partial g_{\mu i}}{\partial x_i}\right.\\ &\left. + g_{\mu i}\frac{\partial}{\partial x_i}\left(f_i + \sum_j \widehat{D}_{ij}x_j\right)\right]. \qquad (19)\end{aligned}$$

Therefore, the effective Fokker-Planck equation in the form of the Kramers-Moyal expansion reads

$$\frac{\partial}{\partial t}P = \sum_i \frac{\partial}{\partial x_i}\left[-\mathcal{D}_1(x_i) + \frac{\partial}{\partial x_i}\mathcal{D}_2(x_i)\right]P, \qquad (20)$$

here the drift and the diffusion coefficients are given by

$$\begin{aligned}\mathcal{D}_1(x_i) &= f_i + \sum_j \widehat{D}_{ij}x_j + \sum_{n=0}\mathcal{D}_1^{(n)}(x_i),\\ \mathcal{D}_2(x_i) &= \sum_{n=0}\mathcal{D}_2^{(n)}(x_i). \qquad (21)\end{aligned}$$

Integrating equation (20) over all variables, with the exception of x_i, and using the fact that the steady state properties are isotropic and translationally invariant, one obtains the following stationary equation for the one-site probability:

$$0 = \frac{\partial}{\partial x_i}\left[-\mathcal{D}_1(x_i) + \frac{\partial}{\partial x_i}\mathcal{D}_2(x_i)\right]P(x_i). \qquad (22)$$

Here we exploit representations of the mean field theory and for the interaction term we get

$$\sum_j \widehat{D}_{ij}x_j = D(\eta - x_i), \qquad (23)$$

where

$$\eta \equiv \eta(x_i) = \int x_j P(x_j|x_i)\mathrm{d}x_j, \qquad j \in nn(j), \qquad (24)$$

is the steady state conditional average of x_j at a neighboring site $j \in nn(j)$, given the value x_i at site i. The conditional average η is the order parameter for the noise induced phase transition. The value of η can be defined from the self-consistency condition (we drop the subscript i for simplicity of notation)

$$\eta = \int_{-\infty}^{\infty} xP(x;\eta)\mathrm{d}x \equiv \mathcal{F}(\eta), \qquad (25)$$

where $P(x;\eta)$ is the solution of the time-independent Fokker-Planck equation. According to equation (22) the stationary distribution takes the form

$$P(x,\eta) = \mathcal{Z}^{-1}(\eta)\mathcal{D}_2^{-1}(x,\eta)\exp\left(\int_0^x \frac{\mathcal{D}_1(x',\eta)}{\mathcal{D}_2(x',\eta)}\mathrm{d}x'\right), \qquad (26)$$

where

$$\mathcal{Z}(\eta) = \int_{-\infty}^{\infty}\mathrm{d}x\mathcal{D}_2^{-1}(x,\eta)\exp\left(\int_0^x \frac{\mathcal{D}_1(x',\eta)}{\mathcal{D}_2(x',\eta)}\mathrm{d}x'\right). \qquad (27)$$

In order to study noise induced phase transitions the solutions of the self-consistent equation (25) and equation of the phase diagram

$$\left.\frac{\mathrm{d}\mathcal{F}(\eta)}{\mathrm{d}\eta}\right|_{\eta=0} = 1. \qquad (28)$$

should be considered. Equation (28) has always a root $\eta = 0$. Nontrivial roots differ only in sign for $\mathrm{d}F/\mathrm{d}\eta|_{\eta=0} > 1$.

Three-parameter synergetic model

We apply the presented method to consider the behavior of generalized Lorenz-type model with noise of each of degrees of freedom:

$$\begin{aligned}\tau_x\dot{x} &= -x + \gamma h + D\Delta x + \zeta_x(\mathbf{r},t),\\ \tau_h\dot{h} &= -h + a_h x\epsilon + \zeta_h(\mathbf{r},t),\\ \tau_\epsilon\dot{\epsilon} &= \epsilon_0 - \epsilon - a_\epsilon xh + \zeta_\epsilon(\mathbf{r},t). \qquad (29)\end{aligned}$$

Here, the first terms on the right-hand sides reflect the autonomous relaxation of the quantities order parameter x, conjugate field h and control parameter ϵ to their respective stationary values $x = 0$, $h = 0$ and $\epsilon = \epsilon_0$; τ_x, τ_h and τ_ϵ are related relaxation times; the positive constants γ, a_h, a_ϵ are the measures of feedbacks; D is the coupling constant. Stochastic terms ζ_x, ζ_h and ζ_ϵ account internal fluctuations which are Gaussian distributed with for exponentially decaying temporal correlations

$$C_{\mu,\nu}(t,t') = \frac{\sigma_\mu\sigma_\nu}{\tau_{\mu,\nu}} \exp\left(-\frac{|t-t'|}{\tau_{\mu,\nu}}\right), \tag{30}$$

where $\mu, \nu = \{x, h, \epsilon\}$. These arise as solutions of an uncoupled set of Langevin equations

$$\tau_\mu \dot{\zeta}_\mu = -\zeta_\mu + \sigma_\mu \xi_\mu(\mathbf{r}, t) \tag{31}$$

with white noise $\xi_\mu(\mathbf{r}, t)$ whose amplitudes are σ_μ.

It is principally important that presented system manifests the Le Chatelier principle: since self-organization is caused by the growth of the control parameter ϵ, then the order parameter x and conjugate field h have to vary in such a way to resist the growth of ϵ. This is caused by the negative feedback between x and h. The positive feedback between x and ϵ leads to an increase in the conjugate field h and is the reason for self-organization of the system. The homogeneous noiseless system (29) was proposed initially to describe the instabilities in laser systems (see [16]); it appeared later to describe exhaustively phase transitions in synergetic systems [17]; the noisy model was proposed quite recently to present the self-organized criticality [18].

In a general case we can define the slow and fast modes to apply to an adiabatic elimination procedure. Usually, the order parameter appears to be the slow variable, therefore, we can set the condition $\tau_x \gg \tau_h, \tau_\epsilon$. Using the second and third equations (29) we can write the conjugate field and control parameter as simple algebraic functions of the order parameter [17]. As a result, we get the equation of motion for the slow variable in the form

$$\dot{x} = f(x) + D\Delta x + \sum_\mu \sigma_\mu g_\mu \zeta_\mu(\mathbf{r}, t), \tag{32}$$

where the deterministic force $f(x) = -\partial V(x)/\partial x$ and noise amplitudes are given by the following equalities:

$$V(x) = \frac{1}{2}\left[x^2 - \theta \ln(1+x^2)\right], \quad \theta \equiv \frac{\epsilon_0}{\gamma\tau_x a_h} \tag{33}$$

$$g_x = 1, \qquad g_h = (1+x^2)^{-1}, \qquad g_\epsilon = x g_h. \tag{34}$$

So, the generalized Lorenz scheme allows one to generate a nontrivial model of stochastic system in the simplest way. In the limit $x \ll 1$, the potential $V(x)$ may be expanded into the Landau-like x^4-potential, whereas the multiplicative functions g_ϵ, g_h take the linear form at constant values of g_x.

Our further approach follows the mean field approximation which replaces the term defining ultimate acts of interactions with the effective interaction force:

$$f_{int}(x, \eta) \equiv D(\eta - x) \tag{35}$$

where η is the order parameter that we define according to the self-consistent equation (25).

For the sake of mathematical simplicity, we consider two non-zeroth noises only: the fluctuations of the slow variable x and the same of the control parameter ϵ (it is easy to see such a choice generalizes some particular cases). Moreover, to reduce the number of parameters, we assume the additive noise to be weak-coloured, *i.e.*

$$C^{(0)}_{x,x} = \sigma_x^2, \quad C^{(1)}_{x,x} = 0. \tag{36}$$

For the multiplicative noise, nontrivial moments are as follows:

$$C^{(0)}_{\epsilon,\epsilon} = \sigma_\epsilon^2, \quad C^{(1)}_{\epsilon,\epsilon} = \tau\sigma_\epsilon^2, \quad \tau \equiv \tau_{\epsilon,\epsilon}. \tag{37}$$

Respectively, moments of the corresponding cross correlation function read:

$$C^{(0)}_{x,\epsilon} = \sigma_x\sigma_\epsilon, \quad C^{(1)}_{x,\epsilon} = \tau_c\sigma_x\sigma_\epsilon, \quad \tau_c \equiv \tau_{x,\epsilon}. \tag{38}$$

Next, we consider, within mean field approach, the noise influence on the system behavior in the cases of both uncorrelated and correlated fluctuations.

Uncorrelated fluctuations

In this case, we start with $C_{x,\epsilon}(t,t') = \sigma_x\sigma_\epsilon\delta(t-t')$. In order to suppress correlations defined through the Novikov theorem [11] we put $C^{(0)}_{x,\epsilon} = 0$ that derives to the standard Fokker-Planck equation with independent noises. Thereby, the terms appearing in equation (20) can be computed to be

$$\begin{aligned} \mathcal{D}_1^{(0)} &= \sigma_\epsilon^2 g_\epsilon \frac{\partial g_\epsilon}{\partial x}, \\ \mathcal{D}_1^{(1)} &= \sigma_\epsilon^2 \tau \frac{\partial g_\epsilon}{\partial x}\frac{\partial}{\partial x} g_\epsilon \left(f + f_{int}\right), \end{aligned} \tag{39}$$

$$\begin{aligned} \mathcal{D}_2^{(0)} &= \sigma_x^2 + \sigma_\epsilon^2 g_\epsilon^2, \\ \mathcal{D}_2^{(1)} &= \sigma_\epsilon^2 \tau g_\epsilon \frac{\partial}{\partial x} g_\epsilon \left(f + f_{int}\right). \end{aligned} \tag{40}$$

Inserting $\mathcal{D}_1$, $\mathcal{D}_2$ into equation (26) and then the result into equation (28), we get an equation for the phase diagram. A related solution is shown in Figure 1 where letters "O" and "D" denote the domains of ordered and disordered phases. The ordered phase is seen to be realized at large values of the control parameter θ, so that the noise induces a transition of a second order from the ordered phase to disordered with its intensity increase. Another conclusion is that the system is ordered at large values of the coupling constant D, and becomes disordered when parameter D decreases. Thus, we arrive at the standard synergetic picture of phase transition where the system becomes ordered when the control parameter increases. As shown in Figure 2, the order parameter η increases monotonically beyond the critical magnitudes of θ and D.

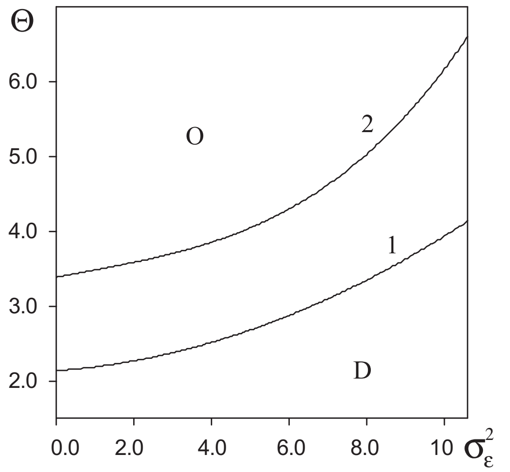

Fig. 1. Phase diagram at $\sigma_x^2 = 1.0$, $\tau = 0.01$: curves 1, 2 correspond to $D = 0.9$, $D = 0.5$.

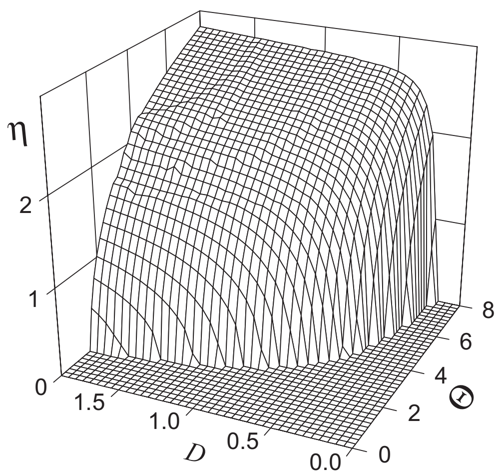

Fig. 2. Order parameter η *vs.* the control parameter θ and the coupling constant D at $\sigma_x = \sigma_\epsilon = 1.0$, $\tau = 0.01$.

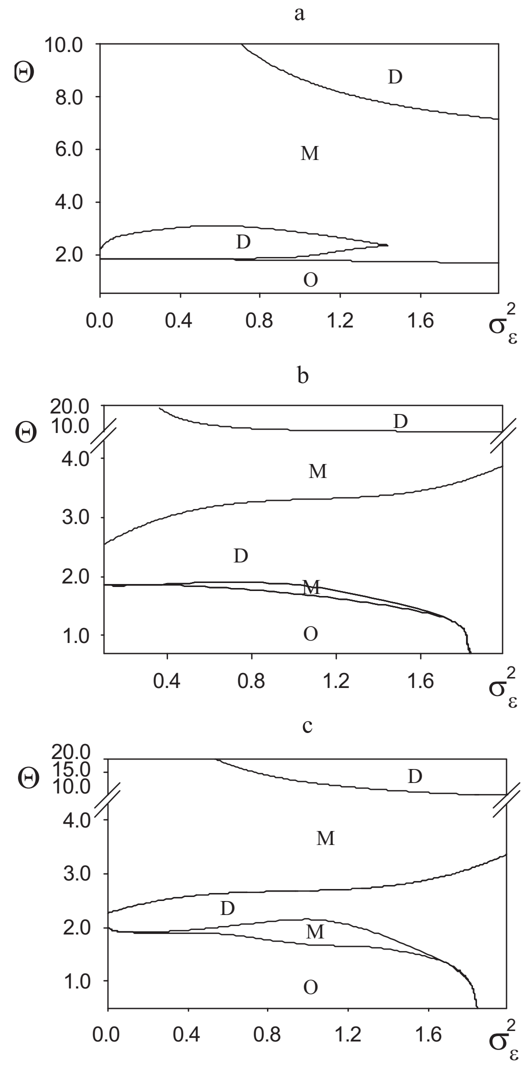

Fig. 3. Phase diagram at $\sigma_x = 1.0$, $\tau = 0.01$: a) $D = 0.9$, $\tau_c = 0.05$; b)$D = 0.9$, $\tau_c = 0.01$; c) $D = 0.95$, $\tau_c = 0.01$.

Effect of cross correlations

The most interesting picture of noise induced phase transition can be found when the noises are correlated. Taking into account terms in equations (18, 19) with $\tau_c \neq 0$, we receive

$$\mathcal{D}_1^{(0)} = \sigma_\epsilon(\sigma_x + \sigma_\epsilon g_\epsilon)\frac{\partial g_\epsilon}{\partial x},$$
$$\mathcal{D}_1^{(1)} = \sigma_\epsilon \frac{\partial g_\epsilon}{\partial x}\frac{\partial}{\partial x}\left[(\tau_c\sigma_x + \tau\sigma_\epsilon g_\epsilon)(f + f_{int})\right], \qquad (41)$$

$$\mathcal{D}_2^{(0)} = \sigma_\epsilon^2[\sigma_x\sigma_\epsilon^{-1}(\sigma_x\sigma_\epsilon^{-1} + 2g_\epsilon) + g_\epsilon^2],$$
$$\mathcal{D}_2^{(1)} = \sigma_\epsilon^2\left[\sigma_x\sigma_\epsilon^{-1}\tau_c\left(\frac{\partial}{\partial x}g_\epsilon + g_\epsilon\frac{\partial}{\partial x}\right)(f + f_{int}) + \tau g_\epsilon\frac{\partial}{\partial x}g_\epsilon(f + f_{int})\right]. \qquad (42)$$

Then, the related equation for the phase diagram arrives at the picture shown in Figure 3 where the control parameter θ is built as the function of the multiplicative noise intensity σ_ϵ^2 at different values of the coupling constant D and the noise cross correlation time τ_c. Here, we show change of the domain forms related to stable ordered (O), disordered (D) and metastable (M) ordered phases. A characteristic peculiarity of the obtained phase diagrams is the appearance of the reentrant transitions with respect to variation of the control parameter θ and the noise intensity σ_ϵ^2: for a range of values of D and τ_c, an ordered state can exist only within intervals of magnitudes θ and σ_ϵ bounded from below and from above. By this way, the picture of phase transitions is enriched by the appearance of a metastable domain being the artefact of the first order phase transitions. The main peculiarities of the system's behavior are as follows. (i) An increase in the cross correlation time τ_c is accompanied by extending the domain of the metastable phase (see Figs. 3a, b). (ii) The size of the domain of the disordered phase decreases when the values of intensity of spatial coupling D increases

(Figs. 3b, c). According to Figure 3a, at small magnitudes σ_ϵ the disordering phase transition is reentrant with respect to θ, where a bifurcation at small θ corresponds to a second order transition, whereas the next bifurcation relates to the first order one. On the other hand, at large values of σ_ϵ we have a chain of first order phase transitions only. For large values τ_c, the disordering reentrant phase transitions of the first order are realized on variation of the parameter σ_ϵ if the control parameter is located in the window $[\theta_1, \theta_2]$.

It is principally important to note that the Figures 3b, c show that at small τ_c two domains of metastable states exist. The bifurcation with respect to variation of the control parameter θ which corresponds to the second order phase transition occurs at both small and large values of σ_ϵ. At intermediate values of the noise intensity, the order of the reentrant phase transition is transformed into the first order. At small values θ the phase transitions with respect to variation of σ_ϵ are of the second order, whereas at large θ such transitions transform to the first order due to the domain appearance of the metastable state. At intermediate values of θ a phase transition disappears and the system is disordered always.

According to the self-consistency equation, the order parameter η varies in dependence of the control parameter θ as is plotted in Figure 4 for different values of σ_ϵ.

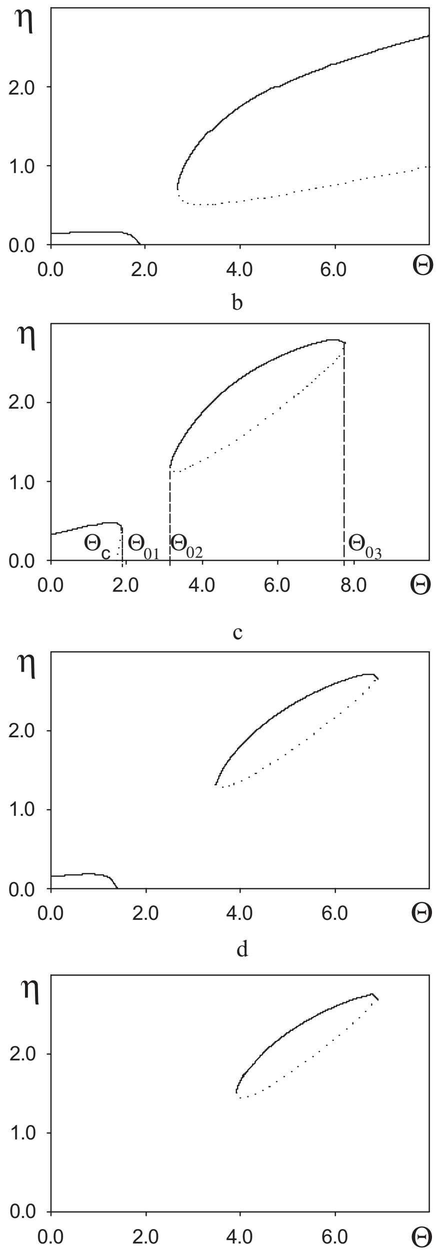

Fig. 4. Order parameter *vs.* the control parameter at $\sigma_x = 1.0$, $\tau = \tau_c = 0.01$, $D = 0.9$: a) $\sigma_\epsilon^2 = 0.09$; b) $\sigma_\epsilon^2 = 1.0$; c) $\sigma_\epsilon^2 = 1.69$; d) $\sigma_\epsilon^2 = 1.96$.

Such a behavior is determined by the main contribution into equation (25) given by the zero-order term $C_{x,\epsilon}^{(0)}$. Let the additive noise intensity be fixed, whereas the intensity of the multiplicative noise of the control parameter takes on increasing values. Then, at small σ_ϵ (Fig. 4a) the order parameter takes a nontrivial magnitude at small values of the control parameter where the ordered phase is stable. A further increase in θ brings the system to disorder by means of a second order phase transition. Due to the bifurcation, there are two positive branches of the solution of equation (25) at large values of θ. Here, the system undergoes a transition of the first order. The double reentrance is observed when we increase the intensity of the multiplicative noise (Fig. 4b). Here, at precritical values $\theta < \theta_c$ there is a stable ordered state only; a further increase of the parameter θ results in the appearance of a branch of unstable state which exists till $\theta < \theta_{01}$. At $\theta_{01} < \theta < \theta_{02}$ the system is disordered. A discontinuous transition toward order is observed at $\theta = \theta_{02}$ and the order parameter has a nontrivial magnitude till $\theta < \theta_{03}$. An increase in the noise intensity of the control parameter transforms a discontinuous transition at small θ to a continuous one (Fig. 4c). Finally, further increase in the noise intensity suppresses the ordering processes at small values of θ keeping the discontinuous ordering within intermediate domain $\theta_{02} < \theta < \theta_{03}$ (Fig. 4d).

The principal peculiarity of the presented picture of a reentrant phase transition is that it appears only within the window $D \simeq 0.85 \div 0.95$ of the coupling constant ($\sigma_x \simeq 1.0$, $\tau = \tau_c \simeq 10^{-2}$). The size of this domain depends on the noise intensities and cross correlation time τ_c. Out of this domain, at small values D, the second order phase transition is realized (Fig. 5a), whereas at large D one has the first order one (Fig. 5b). Such a picture is opposite to the system behavior in the case of uncorrelated fluctuations. Indeed, in accordance with above noted, the situation of uncorrelated noise corresponds to the standard synergetic approach of the phase transition, whereas the case of correlated noise is in good correspondence with a thermodynamic approach where an increase in the control parameter, being a generalized temperature, destroys the ordered phase. Details of such behavior are as follows. First, the noise cross correlation inspires the ordering processes at small and large θ only. Second, an increase in the noise intensity destroys the ordered phase at small θ.

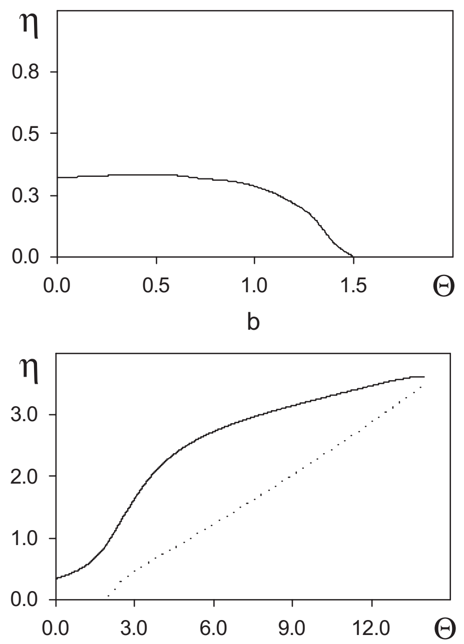

Fig. 5. Order parameter *vs.* the control parameter at $\sigma_x = \sigma_\epsilon = 1.0$, $\tau = \tau_c$: a) $D = 0.5$; b) $D = 1.2$.

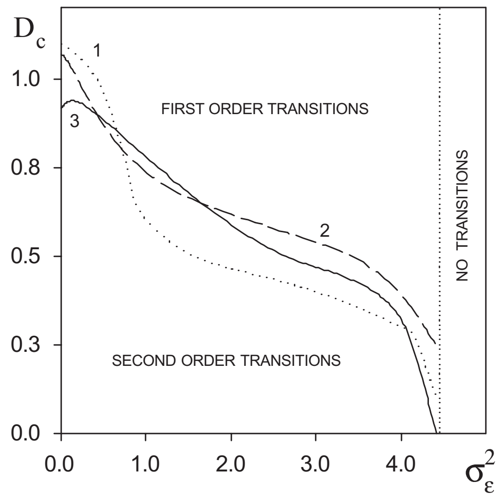

Fig. 6. Phase diagram of transformation of the order noise-induced phase transitions at $\sigma_x = 1.0$: $1 - \tau = 0.05$, $\tau_c = 0.1$; $2 - \tau = \tau_c = 0.05$; $3 - \tau = 0.001$, $\tau_c = 0.05$.

Third, the role of the coupling constant D is to suppress the disordered phase at intermediate values of the control parameter. Therefore, processes of spatial coupling are in competition with noise influence. If contributions of both the spatial coupling and fluctuations are comparable, we get the picture of reentrant noise induced phase transition presented above.

The cross correlation of noises of the order and control parameters is shown to transform the second order phase transition into the first one. Then, the question arises: at what magnitudes of the coupling constant and the control parameter noise intensity the phase transition is of a first order? In Figure 6 we plot the relevant phase diagram for small values θ (at large control parameter θ, only the first order transitions are realized). It can be seen that phase transitions of the second order are realized at small values of both parameters D and σ_ϵ^2. Within both domains of small values σ_ϵ^2 (at large D) and large σ_ϵ^2 (at small D), an increase in the coupling constant leads to the first order phase transitions, whereas an increase in the cross correlation time causes the second order one. The critical magnitude of the coupling constant decreases when the autocorrelation time of the multiplicative noise increases.

Conclusion

This work has focused on the effect of cross correlation between fluctuations and devoted to study the role of such correlation. Through the use of the cumulant expansion method and mean field approximation we have derived the effective Fokker-Planck equation to describe the statistical properties of the system with correlated fluctuations.

The main results are as follows: (i) cross correlation between noise of different degrees of freedom cause the chain of reentrant phase transitions of the first order; (ii) such cross correlation changes the order of phase transitions; (iii) the reentrance effect is caused by the competition between the noise correlation and the spatial coupling; (iv) uncorrelated noises are relevant to the second order phase transition inherent in the synergetics; (v) cross correlation between noise leads to phase transitions related to the thermodynamics.

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